Extensions of solvable potentials with finitely many discrete eigenstates

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Abstract

We address the problem of rational extensions of six examples of shape-invariant potentials having finitely many discrete eigenstates. The *overshoot eigenfunctions* are proposed as candidates unique in this group for the *virtual state wavefunctions*, which are an essential ingredient for *multi-indexed* and *iso-spectral* extensions of these potentials. They have exactly the same form as the eigenfunctions but their degrees are much higher than $n_{\max}$ so that their energies are lower than the groundstate energy.

1 Introduction

Study of exactly solvable potentials in one dimensional quantum mechanics is a very rapidly developing subject in theoretical/mathematical physics in recent years. The main focus is on solvable extensions (deformations) of known exactly solvable potentials [1, 2], in particular, the radial oscillator potential and the Darboux-Pöschl-Teller potential. These potentials have infinitely many discrete eigenstates and the corresponding eigenfunctions consist of classical orthogonal polynomials, the Laguerre and the Jacobi polynomials [3]. Their iso-spectral extensions also produce complete sets of orthogonal polynomials, generically known as multi-indexed Laguerre and Jacobi polynomials [4, 5]–[10] and their various special cases; the simplest single-indexed cases are called exceptional orthogonal polynomials [11]–[36]. The extensions (deformations) of these potentials are achieved by Darboux-Crum transformations

[37, 38] and its modification by Krein-Adler [39]–[41] through the Wronskian formulas (2.6)–(2.8) in terms of *virtual state wavefunctions*, see the definition in section 2. In these cases, the virtual state wavefunctions are obtained from the eigenfunctions by twisting the parameters specified by the *discrete symmetries* of the Hamiltonians (potentials), see (19)–(22) of [4]. Thus these virtual state wavefunctions are also polynomial solutions.

In this paper we will formulate the general procedures of the extensions (deformations) of solvable potentials with *finitely many discrete eigenstates*, which are labeled by the degrees of the polynomial eigenfunctions, $n = 0, 1, \ldots, n_{\max}$. We discuss six explicit examples; Morse potential (M) §4.1, the soliton potential (s) §4.2, Rosen-Morse potential (RM) §4.3, the hyperbolic symmetric top II (hst) §4.4, Kepler problem in hyperbolic space (Eckart potential) (Kh) §4.5 and the hyperbolic Darboux-Pöschl-Teller potential (hDPT) §4.6, according to the naming of the review by Infeld-Hull [1]. The essential point is again the identification of the virtual state wavefunctions, since Darboux-Crum transformations apply equally well in these cases. The discrete symmetries of these potentials, except for (hst) and (hDPT), produce only the *pseudo virtual state wavefunctions* and they provide the known type of non isospectral extensions that are realised by the deletion of eigenstates a lá Krein-Adler [39] with shifted parameters [42]–[47]. This correspondence was proven in detail in [42] for the above six potentials as well as other five well-known potentials having infinitely many discrete eigenstates.

Good candidates for the *virtual states for those potentials having only finitely many discrete eigenstates* are provided by the same polynomial wavefunctions as the eigenfunctions with higher degrees than the highest energy eigenfunction $n > n_{\max}$, so that their energies are lower than the ground state energy. See Figure 1. They are tentatively called *overshoot eigenfunctions*. In her pioneering works [48, 49], Quesne used overshoot eigenfunctions for the "first order SUSY" (single-indexed) extensions of Morse, Rosen-Morse and Eckart potentials. See also [50] in this connection. Once the virtual state wavefunctions are identified, the multi-indexed extensions are trivially realised in the manner explained in [4]. For some potentials *the overshoot eigenfunctions provide a new type of pseudo virtual state wavefunctions*, which generate extensions by adding one eigenstate below the groundstate. These extensions are a new type and they cannot be realised by the deletion of eigenstates a lá Krein-Adler for generic parameters. It can be easily seen that the energy of the added state does not correspond to that of a *negative degree state* except for half integer coupling constant cases.

Extended potentials in terms of virtual state wavefunctions do inherit the shape-invariance [51] of the original potentials as in the cases of the multi-indexed extensions [4]. One new feature of the shape-invariance of the potentials with finitely many discrete eigenstates (except for the hyperbolic Darboux-Pöschl-Teller potential) is that the degrees of the virtual state wavefunctions also shift together with $n_{\max} \to n_{\max} - 1$ as emphasised in Quesne's papers [48, 49]. Similar phenomena have been remarked in the extensions in terms of pseudo virtual states [42].

This paper is organised as follows. The concepts of the virtual and pseudo virtual state wavefunctions are explained briefly in section 2 and the Darboux-Crum transformation formulas in terms of multiple (pseudo) virtual state wavefunctions are recapitulated. The idea of the overshoot eigenfunctions is introduced in section 3 together with the illustration of three types of the energy curves of the systems having finitely many discrete eigenstates. Section 4 is the main part of the paper. Various examples of the virtual state wavefunctions and the new type of pseudo virtual wave functions obtained from the overshoot eigenfunctions are discussed in some detail for the six potentials having finitely many discrete eigenstates. The final section is for a summary and comments.

2 Virtual and Pseudo Virtual States

Here we recapitulate the concepts of virtual state and pseudo virtual state wavefunctions. For more details on the extensions of solvable potentials through Darboux-Crum and Krein-Adler transformations in terms of virtual state and pseudo virtual state wavefunctions, we refer to the recent publication [42].

We consider quantum mechanical systems defined in an interval $x_1 < x < x_2$, which have a finite number of discrete eigenstates with a vanishing groundstate energy,

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x) \ \ (n = 0, 1, \ldots, n_{\max}), \quad 0 = \mathcal{E}_0 < \mathcal{E}_1 < \cdots < \mathcal{E}_{n_{\max}}, \tag{2.1}$$

$$(\phi_m, \phi_n) \overset{\text{def}}{=} \int_{x_1}^{x_2} dx\, \phi_m(x)\phi_n(x) = h_n\delta_{mn} \ \ (h_n > 0). \tag{2.2}$$

The main part of ϕ_n is a certain polynomial. Here are the conditions for the *virtual state wavefunctions* $\tilde{\phi}(x)$, $\mathcal{H}\tilde{\phi}(x) = \tilde{\mathcal{E}}\tilde{\phi}(x)$:

1. No zeros in $x_1 < x < x_2$, *i.e.* $\tilde{\phi}(x) > 0$ or $\tilde{\phi}(x) < 0$ in $x_1 < x < x_2$.

2. Negative energy, $\tilde{\mathcal{E}} < 0$.

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3. $\tilde{\phi}$ is also a polynomial type solution.

4. Square non-integrability, $(\tilde{\phi}, \tilde{\phi}) = \infty$.

5. Reciprocal square non-integrability, $(\tilde{\phi}^{-1}, \tilde{\phi}^{-1}) = \infty$.

Of course these conditions are not totally independent. The negative energy condition is necessary for the positivity of the norm as seen from the norm formula (2.7). When the first condition is dropped and the reciprocal is required to be square integrable $(\tilde{\phi}^{-1}, \tilde{\phi}^{-1}) < \infty$, such seed functions are called *pseudo virtual state wavefunctions*. When the system is extended in terms of a pseudo virtual state wavefunction $\tilde{\phi}(x)$, the new Hamiltonian has an extra *eigenstate* $\tilde{\phi}^{-1}(x)$ with the eigenvalue $\tilde{\mathcal{E}}$, *if the new potential is non-singular*. The extra state is below the original groundstate and the extension is no longer iso-spectral. Since the (pseudo) virtual state wavefunctions $\tilde{\phi}(x)$ are finite in $x_1 < x < x_2$, the non-square integrability can only be caused by the boundaries. Thus the virtual state wavefunctions belong to either of the following type I and II and the pseudo virtual state wavefunctions belong to type III :

$$\text{Type I}: \quad \int_{x_1}^{x_1+\epsilon} dx\, \tilde{\phi}(x)^2 < \infty, \qquad \int_{x_2-\epsilon}^{x_2} dx\, \tilde{\phi}(x)^2 = \infty,$$

$$\& \quad \int_{x_1}^{x_1+\epsilon} dx\, \tilde{\phi}(x)^{-2} = \infty, \qquad \int_{x_2-\epsilon}^{x_2} dx\, \tilde{\phi}(x)^{-2} < \infty, \tag{2.3}$$

$$\text{Type II}: \quad \int_{x_1}^{x_1+\epsilon} dx\, \tilde{\phi}(x)^2 = \infty, \qquad \int_{x_2-\epsilon}^{x_2} dx\, \tilde{\phi}(x)^2 < \infty,$$

$$\& \quad \int_{x_1}^{x_1+\epsilon} dx\, \tilde{\phi}(x)^{-2} < \infty, \qquad \int_{x_2-\epsilon}^{x_2} dx\, \tilde{\phi}(x)^{-2} = \infty, \tag{2.4}$$

$$\text{Type III}: \quad \int_{x_1}^{x_1+\epsilon} dx\, \tilde{\phi}(x)^2 = \infty \ \text{ or } \int_{x_2-\epsilon}^{x_2} dx\, \tilde{\phi}(x)^2 = \infty,$$

$$\& \quad \int_{x_1}^{x_1+\epsilon} dx\, \tilde{\phi}(x)^{-2} < \infty, \qquad \int_{x_2-\epsilon}^{x_2} dx\, \tilde{\phi}(x)^{-2} < \infty. \tag{2.5}$$

An appropriate modification is needed when $x_2 = +\infty$ and/or $x_1 = -\infty$.

The Darboux-Crum transformations in terms of $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \ldots, d_M\}$ (pseudo) virtual state wavefunctions $(\mathcal{H}\tilde{\phi}_{\mathrm{v}}(x) = \tilde{\mathcal{E}}_{\mathrm{v}}\tilde{\phi}_{\mathrm{v}}(x))$ read:

$$\mathcal{H}^{[M]}\phi_n^{[M]}(x) = \mathcal{E}_n\phi_n^{[M]}(x), \tag{2.6}$$

$$\phi_n^{[M]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}, \phi_n](x)}{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)}, \quad (\phi_m^{[M]}, \phi_n^{[M]}) = \prod_{j=1}^{M}(\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j}) \cdot h_n \delta_{mn}, \tag{2.7}$$

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$$U^{[M]}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log\big|W\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)\big|, \tag{2.8}$$

$$\mathcal{H}^{[M]}\tilde{\phi}_{\mathrm{v}}^{[M]}(x) = \tilde{\mathcal{E}}_{\mathrm{v}}\tilde{\phi}_{\mathrm{v}}^{[M]}(x), \quad \tilde{\phi}_{\mathrm{v}}^{[M]}(x) = \frac{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}, \tilde{\phi}_{\mathrm{v}}](x)}{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)}. \tag{2.9}$$

Note that the differential equations (2.6) and (2.9) hold irrespective of the presence of the nodes of the Wronskian $W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)$. By using the Wronskian identity

$$W\big[W[f_1, f_2, \ldots, f_n, g], W[f_1, f_2, \ldots, f_n, h]\big](x)$$
$$= W[f_1, f_2, \ldots, f_n](x) \cdot W[f_1, f_2, \ldots, f_n, g, h](x) \quad (n \geq 0),$$

and (2.9), we can show the following ($s \geq 1$):

$$\frac{d}{dx} \frac{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_{s-1}}, \tilde{\phi}_{d_s}, \tilde{\phi}_{d_{s+1}}](x)}{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_{s-1}}](x)}$$
$$= (\tilde{\mathcal{E}}_{d_s} - \tilde{\mathcal{E}}_{d_{s+1}}) \frac{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_{s-1}}, \tilde{\phi}_{d_s}](x)}{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_{s-1}}](x)} \frac{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_{s-1}}, \tilde{\phi}_{d_{s+1}}](x)}{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_{s-1}}](x)}. \tag{2.10}$$

For deletion of type I virtual sate wavefunctions only (or type II only), this equation (2.10) implies inductively that $W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)$ has no node in $x_1 < x < x_2$ under the conditions

$$\partial_x^s \tilde{\phi}_{d_j}(x)\big|_{x=x_1 \text{ (or } x_2)} = 0 \quad (s = 0, 1, \ldots, M-1). \tag{2.11}$$

For deletion of both type I and type II virtual state wavefunctions such as in [4] and §4.6, we have to check that $W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_{s-1}}, \tilde{\phi}_{d_s}, \tilde{\phi}_{d_{s+1}}](x)/W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_{s-1}}](x)$ has the same sign at $x = x_1$ and $x = x_2$, for no-nodeness of $W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)$.

3 Overshoot Eigenfunctions

The simplest way to construct the virtual and pseudo virtual wavefunctions is to twist the parameters of the eigenfunctions based on the discrete symmetries of the original shape-invariant Hamiltonians. This automatically guarantee that the obtained (pseudo) virtual wavefunctions are of the same polynomial type as the original eigenfunctions. Among the six examples listed in §4, the discrete symmetries produce only pseudo virtual wavefunctions except for the well-known cases of the hyperbolic DPT [20, 22, 4, 42] and the Eckart potential (Kh) [49, 42].

Here we propose *overshoot eigenfunctions* $\tilde{\phi}_{\mathrm{v}}^{\mathrm{os}}(x)$ as the candidates for new types of virtual state and pseudo virtual eigenfunctions for the six examples of shape-invariant potentials

listed in §4. The overshoot eigenfunctions have exactly the same forms as the eigenfunctions $\tilde{\phi}_v^{\mathrm{os}}(x) \stackrel{\text{def}}{=} \phi_v(x)$ ($\tilde{\mathcal{E}}_v^{\mathrm{os}} = \mathcal{E}_v$). But their degrees are much higher than the highest discrete energy level $n_{\max}$ so that their energies are lower than the groundstate energy, which are chosen zero throughout this paper. The following illustrations of the three types of energy curves $\mathcal{E}_n$ with respect to the degree n would be helpful for understanding (Figure 1). In all the three cases the region (a) corresponds to the finitely many discrete eigenstates. The overshoot eigenfunctions correspond to the region (b), which provides the candidates of virtual state wavefunctions and a new type of pseudo virtual wavefunctions. The region (c) in the first two illustrations and the region (c_1) in the third corresponds to the pseudo virtual state wavefunctions.
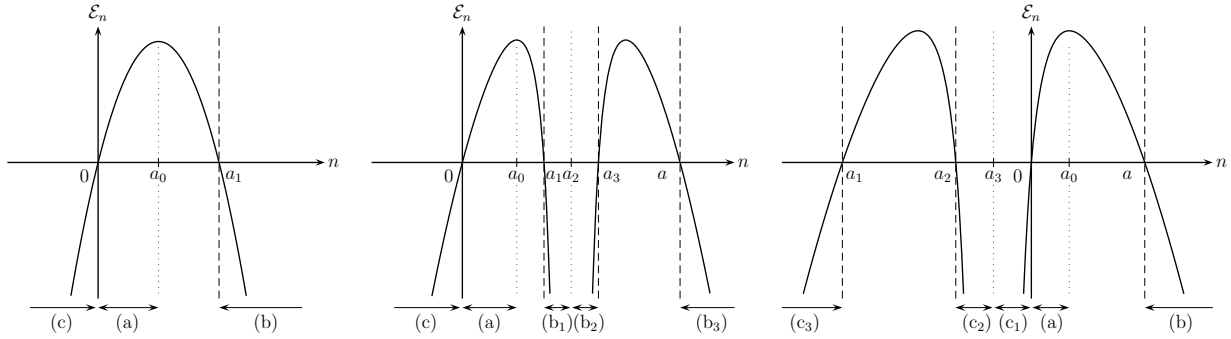

Figure 1: The left represents the energy levels of (M), (s), (hst) and (hDPT). The center corresponds to (RM) and the right to (Kh).

4 Examples of (Pseudo) Virtual State Wavefunctions

Here we provide the explicit forms of the virtual state wavefunctions and a new type of pseudo virtual wavefunctions obtained from the *overshoot eigenfunctions* for six shape-invariant systems §4.1–§4.6, which have finitely many discrete eigenstates. The original systems to be extended usually contain some parameter(s), $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$ and the parameter dependence is denoted by $\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{E}_n(\boldsymbol{\lambda})$, $\phi_n(x; \boldsymbol{\lambda})$, $\tilde{\phi}_n(x; \boldsymbol{\lambda})$, $P_n(\eta(x); \boldsymbol{\lambda})$ etc. As discussed in [42] they are divided into two groups of *eigenfunction patterns*:

$$\text{Group A: (M), (s), (hst), (hDPT),} \quad \phi_n(x; \boldsymbol{\lambda}) = \phi_0(x; \boldsymbol{\lambda}) P_n\big(\eta(x); \boldsymbol{\lambda}\big), \tag{4.1}$$

$$\text{Group B: (K), (RM), (Kh),} \quad \phi_n(x; \boldsymbol{\lambda}) = \phi_0(x; \boldsymbol{\lambda} + n\boldsymbol{\delta}) P_n\big(\eta(x); \boldsymbol{\lambda}\big). \tag{4.2}$$

Here $P_n\big(\eta(x);\boldsymbol{\lambda}\big)$ is a polynomial of degree n in a certain function $\eta(x)$, which is called the sinusoidal coordinate (for Group A) [52]. For these six potentials, except for (Kh) and (hDPT), the twisting of the parameter(s) of the eigenfunctions based on the discrete symmetries of the Hamiltonian generate pseudo virtual states, rather than the virtual states. We will not discuss the discrete symmetries except for (Kh) and (hDPT), since the pseudo virtual states and the rational extensions of these six potentials in terms of them have been explored in detail in [42]. For the parameters of shape-invariant transformation $\boldsymbol{\lambda} \to \boldsymbol{\lambda} + \boldsymbol{\delta}$, we use the symbol g to denote an increasing ($\boldsymbol{\delta} = 1$) parameter and the symbol h for a decreasing ($\boldsymbol{\delta} = -1$) parameter and μ for an unchanging ($\boldsymbol{\delta} = 0$) parameter. Except for §4.7, we assume that the parameter g and/or h take *generic values*, that is not integers or half odd integers. We refer to a recent paper [42] for various properties of these six potentials. No-nodeness of the virtual state wavefunctions can be verified by using the properties of zeros of Laguerre and Jacobi polynomials [3, 45, 32, 49].

The energy of a *new type of pseudo virtual state* $\tilde{\phi}_{\rm v}^{\rm os}(x)$ obtained from the overshoot eigenfunctions does not coincide with the negative energy levels of the original theory except for half integer coupling constant cases. In other words, the previous formula $\tilde{\mathcal{E}}_{\rm v}(\boldsymbol{\lambda}) = \mathcal{E}_{-{\rm v}-1}(\boldsymbol{\lambda})$ in [42] is no longer valid for generic coupling constants. This means that the extensions in terms of the new type of pseudo virtual states cannot be reproduced in terms of the Krein-Adler transformations deleting the (multiple) eigenstates except for half integer coupling constant cases. In other words, the extensions in terms of one pseudo virtual wavefunction of the new type can be made non-singular for some choices of the degrees. But the extensions by using two or more such pseudo virtual wavefunctions would produce singular potentials, since the nodeless condition of the seed functions is no longer guaranteed.

The Wronskian of the eigenfunctions $\phi_n(x;\boldsymbol{\lambda})$ is expressed as [42]

$$\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x;\boldsymbol{\lambda}) = \bar{A}_{\mathcal{D}}(x;\boldsymbol{\lambda})\bar{\Xi}_{\mathcal{D}}\big(\eta(x);\boldsymbol{\lambda}\big), \tag{4.3}$$

$$\bar{A}_{\mathcal{D}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \phi_0(x;\boldsymbol{\lambda})^M \big(c_{\mathcal{F}}^{-1}\frac{d\eta(x)}{dx}\big)^{\frac{1}{2}M(M-1)} & : \text{Group A} \\ \prod_{k=1}^{M} \phi_0(x;\boldsymbol{\lambda} + d_k\boldsymbol{\delta}) & : \text{Group B} \end{cases}, \tag{4.4}$$

$$\bar{\Xi}_{\mathcal{D}}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} c_{\mathcal{F}}^{\frac{1}{2}M(M-1)}\mathrm{W}[P_{d_1}, P_{d_2}, \ldots, P_{d_M}](\eta;\boldsymbol{\lambda}) & : \text{Group A} \\ \det\big(\bar{X}_{j,k}^{\mathcal{D}}(\eta;\boldsymbol{\lambda})\big)_{1\leq j,k\leq M} & : \text{Group B} \end{cases}, \tag{4.5}$$

$$\bar{X}_{j,k}^{\mathcal{D}}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \prod_{i=0}^{j-2} f_{d_k-i}(\boldsymbol{\lambda} + i\boldsymbol{\delta}) \cdot P_{d_k-j+1}\big(\eta;\boldsymbol{\lambda} + (j-1)\boldsymbol{\delta}\big), \tag{4.6}$$

where $c_{\mathcal{F}}$ is -1 (M), 1 (s), 1 (hst) and 4 (hDPT) (See <u>Remark</u> below (3.27) in [42]). The

denominator polynomial $\bar{\Xi}_{\mathcal{D}}(\eta; \boldsymbol{\lambda})$ is a polynomial in η and its degree is generically $\ell_{\mathcal{D}}$:

$$\ell_{\mathcal{D}} \stackrel{\text{def}}{=} \sum_{j=1}^{M} d_j - \frac{1}{2} M(M-1). \tag{4.7}$$

The ratio of the Wronskians becomes

$$\frac{W[\phi_{d_1}, \ldots, \phi_{d_M}, \phi_n](x; \boldsymbol{\lambda})}{W[\phi_{d_1}, \ldots, \phi_{d_M}](x; \boldsymbol{\lambda})} = \frac{\bar{P}_{\mathcal{D},n}(\eta(x); \boldsymbol{\lambda})}{\bar{\Xi}_{\mathcal{D}}(\eta(x); \boldsymbol{\lambda})} \times \begin{cases} \phi_0(x; \boldsymbol{\lambda} + M\boldsymbol{\delta}) & : \text{Group A} \\ \phi_0(x; \boldsymbol{\lambda} + n\boldsymbol{\delta}) & : \text{Group B} \end{cases},$$

$$\bar{P}_{\mathcal{D},n}(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \bar{\Xi}_{\{d_1, \ldots, d_M, n\}}(\eta; \boldsymbol{\lambda}) \quad : \quad \text{degree} = \ell_{\{d_1, \ldots, d_M, n\}} = \ell_{\mathcal{D}} - M + n. \tag{4.8}$$

Since these expressions of the Wronskians are algebraic, they are applicable to the overshoot eigenfunctions $\tilde{\phi}_v^{\text{os}}(x; \boldsymbol{\lambda}) = \phi_v(x; \boldsymbol{\lambda})$ $(\tilde{\mathcal{E}}_v^{\text{os}}(\boldsymbol{\lambda}) = \mathcal{E}_v(\boldsymbol{\lambda}))$.

4.1 Morse potential (M)

The system has finitely many discrete eigenstates $0 \le n \le n_{\max}(\boldsymbol{\lambda}) = [h]'$ in the specified parameter range ($[a]'$ denotes the greatest integer not exceeding and not equal to a):

$$\boldsymbol{\lambda} = (h, \mu), \quad \boldsymbol{\delta} = (-1, 0), \quad -\infty < x < \infty, \quad h, \mu > 0,$$

$$w(x; \boldsymbol{\lambda}) = hx - \mu e^x, \quad U(x; \boldsymbol{\lambda}) = \mu^2 e^{2x} - \mu(2h+1)e^x + h^2,$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = h^2 - (h-n)^2, \quad \eta(x) = e^{-x}, \quad f_n(\boldsymbol{\lambda}) = \frac{n - 2h}{2\mu}, \quad b_{n-1}(\boldsymbol{\lambda}) = -2n\mu,$$

$$\phi_n(x; \boldsymbol{\lambda}) = \phi_0(x; \boldsymbol{\lambda}) P_n(\eta(x); \boldsymbol{\lambda}), \quad \phi_0(x; \boldsymbol{\lambda}) = e^{hx - \mu e^x},$$

$$P_n(\eta; \boldsymbol{\lambda}) = (2\mu\eta^{-1})^{-n} L_n^{(2h-2n)}(2\mu\eta^{-1}), \quad h_n(\boldsymbol{\lambda}) = \frac{\Gamma(2h-n+1)}{(2\mu)^{2h} n! \, 2(h-n)}.$$

4.1.1 virtual states

The energy spectrum is depicted in the left figure ($\mathcal{E}_n(\boldsymbol{\lambda}) < 0$ $(n \ge 0) \Leftrightarrow n > 2h$) of Fig. 1. The overshoot eigenfunctions provide type II virtual state wavefunctions for $v > 2h$ [49]:

$$\tilde{\phi}_v^{\text{os}}(x; \boldsymbol{\lambda}) = \phi_v(x; \boldsymbol{\lambda}), \quad \tilde{\mathcal{E}}_v^{\text{os}}(\boldsymbol{\lambda}) = v(2h - v) \quad (v > 2h). \tag{4.9}$$

Since this type II virtual state wavefunction satisfies the boundary conditions

$$\partial_x^s \tilde{\phi}_v^{\text{os}}(x; \boldsymbol{\lambda}) \big|_{x=\infty} = 0 \quad (s = 0, 1, \ldots), \tag{4.10}$$

a multiple virtual state $\tilde{\phi}_v^{\text{os}}$ deletion gives a non-singular Hamiltonian $\mathcal{H}^{[M]}$. Their 'shape-invariance' relation is

$$w_{\mathcal{D}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \log \left| \frac{W[\tilde{\phi}_{d_1}^{\text{os}}, \ldots, \tilde{\phi}_{d_M}^{\text{os}}, \phi_0](x; \boldsymbol{\lambda})}{W[\tilde{\phi}_{d_1}^{\text{os}}, \ldots, \tilde{\phi}_{d_M}^{\text{os}}](x; \boldsymbol{\lambda})} \right| \quad (\min_j d_j \ge 2),$$

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$$\big(\partial_x w_{\mathcal{D}}(x;\boldsymbol{\lambda})\big)^2 - \partial_x^2 w_{\mathcal{D}}(x;\boldsymbol{\lambda}) = \big(\partial_x w_{\mathcal{D}_-}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)^2 + \partial_x^2 w_{\mathcal{D}_-}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \qquad (4.11)$$

where $\mathcal{D}_- \stackrel{\text{def}}{=} \{d_1 - 1, d_2 - 1, \ldots, d_M - 1\}$ [42].

4.2 Soliton potential (s)

The system has finitely many discrete eigenstates $0 \le n \le n_{\max}(\boldsymbol{\lambda}) = [h]'$ in the specified parameter range:

$$\boldsymbol{\lambda} = h, \quad \boldsymbol{\delta} = -1, \quad -\infty < x < \infty, \quad h > 0,$$

$$w(x;\boldsymbol{\lambda}) = -h\log\cosh x, \quad U(x;\boldsymbol{\lambda}) = -\frac{h(h+1)}{\cosh^2 x} + h^2,$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = h^2 - (h-n)^2, \quad \eta(x) = \sinh x, \quad f_n(\boldsymbol{\lambda}) = h, \quad b_{n-1}(\boldsymbol{\lambda}) = \frac{n(2h-n)}{h},$$

$$\phi_n(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})P_n\big(\eta(x);\boldsymbol{\lambda}\big), \quad \phi_0(x;\boldsymbol{\lambda}) = (\cosh x)^{-h},$$

$$P_n\big(\eta(x);\boldsymbol{\lambda}\big) = (\cosh x)^n P_n^{(h-n,h-n)}(\tanh x), \quad h_n(\boldsymbol{\lambda}) = \frac{2^{2h-2n}\Gamma(h+1)^2}{n!\,(h-n)\Gamma(2h-n+1)}.$$

One can rewrite $P_n(\eta;\boldsymbol{\lambda})$ as

$$P_n(\eta;\boldsymbol{\lambda}) = \frac{(h - [\frac{n-1}{2}])_{[\frac{n+1}{2}]}}{(h - n + \frac{1}{2})_{[\frac{n+1}{2}]}} i^n P_n^{(-h-\frac{1}{2},-h-\frac{1}{2})}(i\eta),$$

where $[a]$ denotes the greatest integer not exceeding a. Note that if we take $\eta(x) = \tanh x$, the eigenfunction has the form of Group B, $\phi_n(x;\boldsymbol{\lambda}) = (\cosh x)^{-h+n} \times P_n^{(h-n,h-n)}(\tanh x)$.

4.2.1 pseudo virtual states

The energy spectrum is depicted in the left figure $(\mathcal{E}_n(\boldsymbol{\lambda}) < 0\ (n \ge 0) \Leftrightarrow n > 2h)$ of Fig. 1. The overshoot eigenfunctions provide a new type of pseudo virtual state wavefunctions for $\text{v} > 2h$:

$$\tilde{\phi}_{\text{v}}^{\text{os}}(x;\boldsymbol{\lambda}) = \phi_{\text{v}}(x;\boldsymbol{\lambda}), \quad \tilde{\mathcal{E}}_{\text{v}}^{\text{os}}(\boldsymbol{\lambda}) = \text{v}(2h - \text{v}) \quad (\text{v} > 2h). \qquad (4.12)$$

In the absence of virtual state wavefunctions, shape-invariant rational extensions of the soliton potential are not possible.

4.3 Rosen-Morse potential (RM)

This potential is also called Rosen-Morse II potential. The system has finitely many discrete eigenstates $0 \le n \le n_{\max}(\boldsymbol{\lambda}) = [h - \sqrt{\mu}]'$ in the specified parameter range:

$$\boldsymbol{\lambda} = (h, \mu), \quad \boldsymbol{\delta} = (-1, 0), \quad -\infty < x < \infty, \quad h > \sqrt{\mu} > 0,$$

$$w(x;\boldsymbol{\lambda}) = -h\log\cosh x - \frac{\mu}{h}x, \quad U(x;\boldsymbol{\lambda}) = -\frac{h(h+1)}{\cosh^2 x} + 2\mu\tanh x + h^2 + \frac{\mu^2}{h^2},$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = h^2 - (h-n)^2 + \frac{\mu^2}{h^2} - \frac{\mu^2}{(h-n)^2}, \quad \eta(x) = \tanh x,$$

$$f_n(\boldsymbol{\lambda}) = \frac{h^2(h-n)^2 - \mu^2}{h(h-n)^2}, \quad b_{n-1}(\boldsymbol{\lambda}) = \frac{n(2h-n)}{h},$$

$$\phi_n(x;\boldsymbol{\lambda}) = e^{-\frac{\mu}{h-n}x}(\cosh x)^{-h+n}P_n\big(\eta(x);\boldsymbol{\lambda}\big), \quad \phi_0(x;\boldsymbol{\lambda}) = e^{-\frac{\mu}{h}x}(\cosh x)^{-h},$$

$$P_n(\eta;\boldsymbol{\lambda}) = P_n^{(\alpha_n,\beta_n)}(\eta), \quad \alpha_n = h - n + \frac{\mu}{h-n}, \quad \beta_n = h - n - \frac{\mu}{h-n},$$

$$h_n(\boldsymbol{\lambda}) = \frac{2^{2h-2n}(h-n)\Gamma(h + \frac{\mu}{h-n} + 1)\Gamma(h - \frac{\mu}{h-n} + 1)}{n!\left((h-n)^2 - \frac{\mu^2}{(h-n)^2}\right)\Gamma(2h-n+1)}.$$

4.3.1 virtual states and pseudo virtual states

The energy spectrum is depicted in the center figure $(\mathcal{E}_n(\boldsymbol{\lambda}) < 0 \; (n \geq 0) \Leftrightarrow h - \frac{\mu}{h} < n < h$ or $h < n < h + \frac{\mu}{h}$ or $n > 2h)$ of Fig. 1. The overshoot eigenfunctions provide two types of virtual state wavefunctions [49] and a new type of pseudo virtual wavefunctions:

$$\tilde{\phi}_{\mathrm{v}}^{\mathrm{os}}(x;\boldsymbol{\lambda}) = \phi_{\mathrm{v}}(x;\boldsymbol{\lambda}), \quad \tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{os}}(\boldsymbol{\lambda}) = \mathrm{v}(2h-\mathrm{v})\frac{(h-\mathrm{v}+\frac{\mu}{h})(h-\mathrm{v}-\frac{\mu}{h})}{(h-\mathrm{v})^2},$$

$$\begin{cases} \text{type II virtual states} \quad : \quad h - \dfrac{\mu}{h} < \mathrm{v} < h \\ \text{type I \; virtual states} \quad : \quad h < \mathrm{v} < h + \dfrac{\mu}{h} \\ \text{pseudo virtual states} \quad : \quad \mathrm{v} > 2h \end{cases} \tag{4.13}$$

The type II virtual states correspond to the region (b_1), whereas the overshoot eigenfunctions in region (b_2) provide type I virtual states. The negative energy condition $\tilde{\mathcal{E}}_{\mathrm{v}} < 0$ is also satisfied in the region (b_3), $\mathrm{v} > 2h$, and in this range the overshoot eigenfunctions provide the new type of pseudo virtual state wavefunctions. Since these virtual state wavefunctions satisfy the boundary conditions

$$\text{type II virtual state}: \quad \partial_x^s \tilde{\phi}_{\mathrm{v}}^{\mathrm{os}}(x;\boldsymbol{\lambda})\big|_{x=\infty} = 0 \quad (s=0,1,\ldots), \tag{4.14}$$

$$\text{type I virtual state}: \quad \partial_x^s \tilde{\phi}_{\mathrm{v}}^{\mathrm{os}}(x;\boldsymbol{\lambda})\big|_{x=-\infty} = 0 \quad (s=0,1,\ldots), \tag{4.15}$$

the multiple virtual state $\tilde{\phi}_{\mathrm{v}}^{\mathrm{os}}$ deletion (type II only or type I only) gives a non-singular Hamiltonian $\mathcal{H}^{[M]}$. They satisfy the 'shape-invariance' relation (4.11).

4.4 Hyperbolic symmetric top II (hst)

The system has finitely many discrete eigenstates $0 \le n \le n_{\max}(\boldsymbol{\lambda}) = [h]'$ in the specified parameter range:

$$\boldsymbol{\lambda} = (h, \mu), \quad \boldsymbol{\delta} = (-1, 0), \quad -\infty < x < \infty, \quad h, \mu > 0,$$

$$w(x; \boldsymbol{\lambda}) = -h \log \cosh x - \mu \tan^{-1} \sinh x,$$

$$U(x; \boldsymbol{\lambda}) = \frac{-h(h+1) + \mu^2 + \mu(2h+1) \sinh x}{\cosh^2 x} + h^2,$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = h^2 - (h - n)^2, \quad \eta(x) = \sinh x, \quad f_n(\boldsymbol{\lambda}) = \frac{n - 2h}{2}, \quad b_{n-1}(\boldsymbol{\lambda}) = -2n,$$

$$\phi_n(x; \boldsymbol{\lambda}) = \phi_0(x; \boldsymbol{\lambda}) P_n(\eta(x); \boldsymbol{\lambda}), \quad \phi_0(x; \boldsymbol{\lambda}) = e^{-\mu \tan^{-1} \sinh x} (\cosh x)^{-h},$$

$$P_n(\eta; \boldsymbol{\lambda}) = i^{-n} P_n^{(\alpha, \beta)}(i\eta), \quad \alpha = -h - \tfrac{1}{2} - i\mu, \quad \beta = -h - \tfrac{1}{2} + i\mu,$$

$$h_n(\boldsymbol{\lambda}) = \frac{\pi \Gamma(2h - n + 1)}{2^{2h} n! \, (h - n) \Gamma(h - n + \tfrac{1}{2} + i\mu) \Gamma(h - n + \tfrac{1}{2} - i\mu)}.$$

4.4.1 pseudo virtual states

The energy spectrum is depicted in the left figure $(\mathcal{E}_n(\boldsymbol{\lambda}) < 0 \ (n \ge 0) \Leftrightarrow n > 2h)$ of Fig. 1. The overshoot eigenfunctions in region (b) provide the new type of pseudo virtual state wavefunctions for $v > 2h$:

$$\tilde{\phi}_v^{\text{os}}(x; \boldsymbol{\lambda}) = \phi_v(x; \boldsymbol{\lambda}), \quad \tilde{\mathcal{E}}_v^{\text{os}}(\boldsymbol{\lambda}) = v(2h - v) \quad (v > 2h). \tag{4.16}$$

In the absence of virtual state wavefunctions, shape-invariant rational extensions of the hyperbolic symmetric top II potential are not possible.

4.5 Kepler problem in hyperbolic space (Kh)

This potential is also called Eckart potential. It has finitely many discrete eigenstates $0 \le n \le n_{\max}(\boldsymbol{\lambda}) = [\sqrt{\mu} - g]'$ in the specified parameter range:

$$\boldsymbol{\lambda} = (g, \mu), \quad \boldsymbol{\delta} = (1, 0), \quad 0 < x < \infty, \quad \sqrt{\mu} > g > \frac{1}{2},$$

$$w(x; \boldsymbol{\lambda}) = g \log \sinh x - \frac{\mu}{g} x, \quad U(x; \boldsymbol{\lambda}) = \frac{g(g-1)}{\sinh^2 x} - 2\mu \coth x + g^2 + \frac{\mu^2}{g^2},$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = g^2 - (g + n)^2 + \frac{\mu^2}{g^2} - \frac{\mu^2}{(g + n)^2}, \quad \eta(x) = \coth x,$$

$$f_n(\boldsymbol{\lambda}) = \frac{\mu^2 - g^2(g + n)^2}{g(g + n)^2}, \quad b_{n-1}(\boldsymbol{\lambda}) = \frac{n(2g + n)}{g},$$

$$\phi_n(x;\boldsymbol{\lambda}) = e^{-\frac{\mu}{g+n}x}(\sinh x)^{g+n}P_n\big(\eta(x);\boldsymbol{\lambda}\big), \quad \phi_0(x;\boldsymbol{\lambda}) = e^{-\frac{\mu}{g}x}(\sinh x)^g,$$

$$P_n(\eta;\boldsymbol{\lambda}) = P_n^{(\alpha_n,\beta_n)}(\eta), \quad \alpha_n = -g - n + \frac{\mu}{g+n}, \quad \beta_n = -g - n - \frac{\mu}{g+n},$$

$$h_n(\boldsymbol{\lambda}) = \frac{(g+n)\Gamma(1-g+\frac{\mu}{g+n})\Gamma(2g+n)}{2^{2g+2n}n!\left(\frac{\mu^2}{(g+n)^2}-(g+n)^2\right)\Gamma(g+\frac{\mu}{g+n})}.$$

The discrete symmetry and the pseudo virtual state wavefunctions are:

$$\mathcal{H}(\boldsymbol{\lambda}) = \mathcal{H}\big(\mathfrak{t}(\boldsymbol{\lambda})\big) + \mathcal{E}_{-1}(\boldsymbol{\lambda}), \quad \mathfrak{t}(\boldsymbol{\lambda}) = (1-g,\mu),$$

$$\tilde{\phi}_v(x;\boldsymbol{\lambda}) = \phi_v\big(x;\mathfrak{t}(\boldsymbol{\lambda})\big) = e^{\frac{\mu}{g-v-1}x}(\sinh x)^{-g+v+1}P_v\big(\eta(x);\mathfrak{t}(\boldsymbol{\lambda})\big),$$

$$\tilde{\mathcal{E}}_v(\boldsymbol{\lambda}) = \mathcal{E}_v\big(\mathfrak{t}(\boldsymbol{\lambda})\big) + \mathcal{E}_{-1}(\boldsymbol{\lambda}) = \mathcal{E}_{-v-1}(\boldsymbol{\lambda}) \quad \left(0 \le v < g-1, \; v > \frac{\mu}{g}+g-1\right). \tag{4.17}$$

In the following we restrict to $g > \frac{3}{2}$.

4.5.1 virtual states

The energy spectrum is depicted in the right figure of Fig. 1, $(\mathcal{E}_n(\boldsymbol{\lambda}) < 0 \; (n \ge 0) \Leftrightarrow n > \frac{\mu}{g}-g$, $\tilde{\mathcal{E}}_v(\boldsymbol{\lambda}) < 0 \; (v \ge 0) \Leftrightarrow 0 \le v < g-1$ or $g-1 < v < 2g-1$ or $v > \frac{\mu}{g}+g-1$). In this case the discrete symmetry generates type II virtual state wavefunctions for $g-1 < v < 2g-1$ [49, 42]. This is the only example, except for (hDPT), that the discrete symmetry provides the virtual state wavefunctions. Since this type II virtual state wavefunction satisfies the boundary conditions

$$\partial_x^s \tilde{\phi}_v(x;\boldsymbol{\lambda})\big|_{x=\infty} = 0 \quad (s = 0, 1, \ldots), \tag{4.18}$$

a multiple virtual state $\tilde{\phi}_v$ deletion gives a non-singular Hamiltonian $\mathcal{H}^{[M]}$. The 'shape-invariance' relation is

$$w_{\mathcal{D}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \log\left|\frac{W[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M},\phi_0](x;\boldsymbol{\lambda})}{W[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M}](x;\boldsymbol{\lambda})}\right|,$$

$$\big(\partial_x w_{\mathcal{D}}(x;\boldsymbol{\lambda})\big)^2 - \partial_x^2 w_{\mathcal{D}}(x;\boldsymbol{\lambda}) = \big(\partial_x w_{\mathcal{D}_+}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)^2 + \partial_x^2 w_{\mathcal{D}_+}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \tag{4.19}$$

where $\mathcal{D}_+ \stackrel{\text{def}}{=} \{d_1+1, d_2+1, \ldots, d_M+1\}$ [42].

The overshoot eigenfunctions in region (b) provide type I virtual state wavefunctions for $v > \frac{\mu}{g}-g$ [49]:

$$\tilde{\phi}_v^{\text{os}}(x;\boldsymbol{\lambda}) = \phi_v(x;\boldsymbol{\lambda}),$$

$$\tilde{\mathcal{E}}_v^{\text{os}}(\boldsymbol{\lambda}) = -v(2g+v)\frac{(g+v+\frac{\mu}{g})(g+v-\frac{\mu}{g})}{(g+v)^2} \quad \left(v > \frac{\mu}{g}-g\right). \tag{4.20}$$

Since this type I virtual state wavefunction satisfies the boundary conditions

$$\partial_x^s \tilde{\phi}_{\mathrm{v}}^{\mathrm{os}}(x; \boldsymbol{\lambda}) \big|_{x=0} = 0 \quad (s = 0, 1, \ldots, M-1) \ \text{ for } \ g > M - 1, \tag{4.21}$$

a multiple virtual state $\tilde{\phi}_{\mathrm{v}}^{\mathrm{os}}$ deletion gives a non-singular Hamiltonian $\mathcal{H}^{[M]}$ for $g > M - 1$. It satisfies the 'shape-invariance' relation (4.11).

4.6 Hyperbolic Darboux-Pöschl-Teller potential (hDPT)

It has finitely many discrete eigenstates $0 \leq n \leq n_{\max}(\boldsymbol{\lambda}) = [\frac{h-g}{2}]'$ in the specified parameter range:

$$\boldsymbol{\lambda} = (g, h), \quad \boldsymbol{\delta} = (1, -1), \quad 0 < x < \infty, \quad h > g > \frac{1}{2},$$

$$w(x; \boldsymbol{\lambda}) = g \log \sinh x - h \log \cosh x, \quad U(x; \boldsymbol{\lambda}) = \frac{g(g-1)}{\sinh^2 x} - \frac{h(h+1)}{\cosh^2 x} + (h - g)^2,$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4n(h - g - n), \quad \eta(x) = \cosh 2x, \quad f_n(\boldsymbol{\lambda}) = 2(n + g - h), \quad b_{n-1}(\boldsymbol{\lambda}) = -2n,$$

$$\phi_n(x; \boldsymbol{\lambda}) = \phi_0(x; \boldsymbol{\lambda}) P_n(\eta(x); \boldsymbol{\lambda}), \quad \phi_0(x; \boldsymbol{\lambda}) = (\sinh x)^g (\cosh x)^{-h},$$

$$P_n(\eta; \boldsymbol{\lambda}) = P_n^{(g-\frac{1}{2}, -h-\frac{1}{2})}(\eta), \quad h_n(\boldsymbol{\lambda}) = \frac{\Gamma(n + g + \frac{1}{2}) \Gamma(h - g - n + 1)}{2 \, n! \, (h - g - 2n) \Gamma(h - n + \frac{1}{2})}.$$

The eigenvalues can be also expressed as $\mathcal{E}_n(\boldsymbol{\lambda}) = 4\big(\big(\frac{h-g}{2}\big)^2 - \big(\frac{h-g}{2} - n\big)^2\big)$.

Three types of discrete symmetries are:

$$\text{type I}: \quad \mathcal{H}(\boldsymbol{\lambda}) = \mathcal{H}\big(\mathfrak{t}^{\mathrm{I}}(\boldsymbol{\lambda})\big) - (1 + 2g)(1 + 2h), \quad \mathfrak{t}^{\mathrm{I}}(\boldsymbol{\lambda}) = (g, -1 - h),$$

$$\text{type II}: \quad \mathcal{H}(\boldsymbol{\lambda}) = \mathcal{H}\big(\mathfrak{t}^{\mathrm{II}}(\boldsymbol{\lambda})\big) - (1 - 2g)(1 - 2h), \quad \mathfrak{t}^{\mathrm{II}}(\boldsymbol{\lambda}) = (1 - g, h),$$

$$\text{type III}: \quad \mathcal{H}(\boldsymbol{\lambda}) = \mathcal{H}\big(\mathfrak{t}(\boldsymbol{\lambda})\big) + \mathcal{E}_{-1}(\boldsymbol{\lambda}), \quad \mathfrak{t} = \mathfrak{t}^{\mathrm{II}} \circ \mathfrak{t}^{\mathrm{I}}, \quad \mathfrak{t}(\boldsymbol{\lambda}) = (1 - g, -1 - h).$$

The pseudo virtual state wavefunctions are generated by type III symmetry [42]. In the following we restrict to $g > \frac{3}{2}$.

4.6.1 virtual states

The energy spectrum is depicted in the left figure $(\mathcal{E}_n(\boldsymbol{\lambda}) < 0 \ (n \geq 0) \Leftrightarrow n > h - g)$ of Fig. 1. The overshoot eigenfunctions in region (b) provide type I virtual state wavefunctions for $\mathrm{v} > h - g$:

$$\tilde{\phi}_{\mathrm{v}}^{\mathrm{os}}(x; \boldsymbol{\lambda}) = \phi_{\mathrm{v}}(x; \boldsymbol{\lambda}), \quad \tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{os}}(\boldsymbol{\lambda}) = -4\mathrm{v}(g - h + \mathrm{v}) \quad (\mathrm{v} > h - g). \tag{4.22}$$

Since this type I virtual state wavefunction satisfies the boundary conditions

$$\partial_x^s \tilde{\phi}_{\rm v}^{\rm os}(x;\boldsymbol{\lambda})\big|_{x=0} = 0 \quad (s = 0, 1, \ldots, M-1) \ \ \text{for} \ \ g > M-1, \tag{4.23}$$

a multiple virtual state $\tilde{\phi}_{\rm v}^{\rm os}$ deletion gives a non-singular Hamiltonian $\mathcal{H}^{[M]}$ for $g > M-1$. It satisfies the 'shape-invariance' relation (4.11).

In the rest of this subsection we present various formulas for the type I and type II virtual states which are generated by the discrete symmetries of the Hamiltonian [15, 22, 42]:

$$\tilde{\phi}_{\rm v}^{\rm I}(x;\boldsymbol{\lambda}) = \phi_{\rm v}\big(x; {\rm t}^{\rm I}(\boldsymbol{\lambda})\big) = (\sinh x)^g (\cosh x)^{h+1} P_{\rm v}\big(\eta(x); {\rm t}^{\rm I}(\boldsymbol{\lambda})\big) \quad ({\rm v} \geq 0),$$

$$\tilde{\mathcal{E}}_{\rm v}^{\rm I}(\boldsymbol{\lambda}) = \mathcal{E}_{\rm v}\big({\rm t}^{\rm I}(\boldsymbol{\lambda})\big) - (1+2g)(1+2h) = -(2{\rm v}+1+2g)(2{\rm v}+1+2h), \tag{4.24}$$

$$\tilde{\phi}_{\rm v}^{\rm II}(x;\boldsymbol{\lambda}) = \phi_{\rm v}\big(x; {\rm t}^{\rm II}(\boldsymbol{\lambda})\big) = (\sinh x)^{1-g} (\cosh x)^{-h} P_{\rm v}\big(\eta(x); {\rm t}^{\rm II}(\boldsymbol{\lambda})\big) \quad (0 \leq {\rm v} < g - \tfrac{1}{2}),$$

$$\tilde{\mathcal{E}}_{\rm v}^{\rm II}(\boldsymbol{\lambda}) = \mathcal{E}_{\rm v}\big({\rm t}^{\rm II}(\boldsymbol{\lambda})\big) - (1-2g)(1-2h) = -(2{\rm v}+1-2g)(2{\rm v}+1-2h). \tag{4.25}$$

Since these virtual state wavefunctions satisfy the boundary conditions

$$\partial_x^s \tilde{\phi}_{\rm v}^{\rm I}(x;\boldsymbol{\lambda})\big|_{x=0} = 0 \quad (s = 0, 1, \ldots, M-1) \ \ \text{for} \ \ g > M-1, \tag{4.26}$$

$$\partial_x^s \tilde{\phi}_{\rm v}^{\rm II}(x;\boldsymbol{\lambda})\big|_{x=\infty} = 0 \quad (s = 0, 1, \ldots,), \tag{4.27}$$

a multiple virtual state $\tilde{\phi}_{\rm v}^{\rm I}$ (or $\tilde{\phi}_{\rm v}^{\rm II}$) deletion gives a non-singular Hamiltonian $\mathcal{H}^{[M]}$ ($g > M-1$ for type I). For a mixed deletion of type I and II,

$$\mathcal{D} = \{d_1, \ldots, d_M\} = \{d_1^{\rm I}, \ldots, d_{M_{\rm I}}^{\rm I}, d_1^{\rm II}, \ldots, d_{M_{\rm II}}^{\rm II}\} \ \ (M = M_{\rm I} + M_{\rm II}),$$

the Wronskian formulas are

$$\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x;\boldsymbol{\lambda}) = A_{\mathcal{D}}(x;\boldsymbol{\lambda}) \Xi_{\mathcal{D}}\big(\eta(x);\boldsymbol{\lambda}\big), \tag{4.28}$$

$$\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}, \phi_n](x;\boldsymbol{\lambda}) = A_{\mathcal{D},n}(x;\boldsymbol{\lambda}) P_{\mathcal{D},n}\big(\eta(x);\boldsymbol{\lambda}\big), \tag{4.29}$$

$$A_{\mathcal{D}}(x;\boldsymbol{\lambda}) = (\sinh x)^{g(M_{\rm I}-M_{\rm II})+\frac{1}{2}M_{\rm I}(M_{\rm I}-1)+\frac{1}{2}M_{\rm II}(M_{\rm II}+1)-M_{\rm I}M_{\rm II}}$$
$$\times (\cosh x)^{-h(M_{\rm II}-M_{\rm I})+\frac{1}{2}M_{\rm I}(M_{\rm I}+1)+\frac{1}{2}M_{\rm II}(M_{\rm II}-1)-M_{\rm I}M_{\rm II}} \times c_{\mathcal{F}}^{-\frac{1}{2}M(M-1)}, \tag{4.30}$$

$$A_{\mathcal{D},n}(x;\boldsymbol{\lambda}) = (\sinh x)^{g(M_{\rm I}-M_{\rm II}+1)+\frac{1}{2}M_{\rm I}(M_{\rm I}+1)+\frac{1}{2}M_{\rm II}(M_{\rm II}-1)-M_{\rm I}M_{\rm II}}$$
$$\times (\cosh x)^{-h(M_{\rm II}-M_{\rm I}+1)+\frac{1}{2}M_{\rm I}(M_{\rm I}-1)+\frac{1}{2}M_{\rm II}(M_{\rm II}+1)-M_{\rm I}M_{\rm II}} \times c_{\mathcal{F}}^{-\frac{1}{2}M(M+1)}. \tag{4.31}$$

Here $\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})$ and $P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda})$ are polynomials in η and their degrees are generically $\ell'_{\mathcal{D}}$ and $\ell'_{\mathcal{D}} + n$ respectively, where $\ell'_{\mathcal{D}}$ is

$$\ell'_{\mathcal{D}} = \sum_{j=1}^{M_{\rm I}} d_j^{\rm I} + \sum_{j=1}^{M_{\rm II}} d_j^{\rm II} - \frac{1}{2}M_{\rm I}(M_{\rm I}-1) - \frac{1}{2}M_{\rm II}(M_{\rm II}-1) + M_{\rm I}M_{\rm II}. \tag{4.32}$$

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The ratio of the Wronskians becomes

$$\frac{\mathrm{W}[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M},\phi_n](x;\boldsymbol{\lambda})}{\mathrm{W}[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M}](x;\boldsymbol{\lambda})} = \frac{P_{\mathcal{D},n}\big(\eta(x);\boldsymbol{\lambda}\big)}{\Xi_{\mathcal{D}}\big(\eta(x);\boldsymbol{\lambda}\big)} \times \phi_0(x;\boldsymbol{\lambda}+M_{\mathrm{I}}\tilde{\boldsymbol{\delta}}^{\mathrm{I}}+M_{\mathrm{II}}\tilde{\boldsymbol{\delta}}^{\mathrm{II}}), \tag{4.33}$$

where $\tilde{\boldsymbol{\delta}}^{\mathrm{I}} = (1,1)$ and $\tilde{\boldsymbol{\delta}}^{\mathrm{II}} = (-1,-1)$. The lowest degree polynomial $P_{\mathcal{D},0}(\eta;\boldsymbol{\lambda})$ is related to the denominator polynomial $\Xi_{\mathcal{D}}$ with shifted parameters

$$P_{\mathcal{D},0}(\eta;\boldsymbol{\lambda}) \propto \Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}). \tag{4.34}$$

For a proper choice of the parameter range, the Hamiltonian $\mathcal{H}^{[M]}$ is non-singular as in the trigonometric DPT case [4]. The ordinary shape-invariance relation holds:

$$w_{\mathcal{D}}(x;\boldsymbol{\lambda}) = \log\left|\frac{\mathrm{W}[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M},\phi_0](x;\boldsymbol{\lambda})}{\mathrm{W}[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M}](x;\boldsymbol{\lambda})}\right|,$$
$$\big(\partial_x w_{\mathcal{D}}(x;\boldsymbol{\lambda})\big)^2 - \partial_x^2 w_{\mathcal{D}}(x;\boldsymbol{\lambda}) = \big(\partial_x w_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)^2 + \partial_x^2 w_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}). \tag{4.35}$$

4.7   New pseudo virtual states with half integer coupling constant

A Darboux transformation in terms of a pseudo virtual state $(\tilde{\phi}_{\mathrm{v}}(x),\tilde{\mathcal{E}}_{\mathrm{v}})$ creates a new eigenstate with the energy $\tilde{\mathcal{E}}_{\mathrm{v}}$. For the ordinary pseudo virtual states created by the twisting of parameters, the energy is always equal to that of the original 'eigenstate' with a negative degree $\tilde{\mathcal{E}}_{\mathrm{v}} = \mathcal{E}_{-\mathrm{v}-1}$ (4.17), as emphasised in [42]. For the new type of pseudo virtual states found in (s) §4.2, (RM) §4.3 and (hst) §4.4, their energies do not coincide with the negative energy levels of the original system in general. However it happens for half integer coupling constant cases. In these cases the possibility arises that the Darboux-Crum transformations in terms of multiple pseudo virtual states are equivalent to the Krein-Adler transformations in terms of multiple eigenstates with shifted parameters, like those explained in [42]. In fact, this happens for (s) with a half odd integer h and for (RM) with a half integer h.

In (RM), the energy function $\mathcal{E}_n(\boldsymbol{\lambda})$ $(n \in \mathbb{R})$ satisfies $\mathcal{E}_n(\boldsymbol{\lambda}) = \mathcal{E}_{2h-n}(\boldsymbol{\lambda})$. In the following we assume that the coupling constant h is a half integer, $2h \in \mathbb{Z}_{>0}$. Then, for integers v $(\mathrm{v} > 2h)$ and $\mathrm{v}' = \mathrm{v} - 2h - 1 \geq 0$, we have $\mathcal{E}_{\mathrm{v}}(\boldsymbol{\lambda}) = \mathcal{E}_{-\mathrm{v}'-1}(\boldsymbol{\lambda})$, namely $\tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{os}}(\boldsymbol{\lambda}) = \mathcal{E}_{-\mathrm{v}'-1}(\boldsymbol{\lambda})$, which resembles $\tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda}) = \mathcal{E}_{-\mathrm{v}-1}(\boldsymbol{\lambda})$ in [42]. Moreover, it is easy to see that the degree of the polynomial is reduced accordingly,

$$\deg\big(P_{\mathrm{v}}(\eta;\boldsymbol{\lambda})\big) = \mathrm{v} - 2h - 1 = \mathrm{v}' \quad (\mathrm{v} > 2h). \tag{4.36}$$

Let $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \ldots, d_M\}$ $(d_j > 2h)$ be a set of distinct non-negative integers and fix an integer $N \geq \max(\mathcal{D}) - 1$. Let us define a set of distinct non-negative integers $\bar{\mathcal{D}} = \{0, 1, \ldots, N\} \backslash \{\bar{d}'_1, \bar{d}'_2, \ldots, \bar{d}'_M\}$ together with the shifted parameters $\bar{\boldsymbol{\lambda}}$:

$$\bar{\mathcal{D}} \stackrel{\text{def}}{=} \{0, 1, \ldots, \breve{d}'_1, \ldots, \breve{d}'_2, \ldots, \breve{d}'_M, \ldots, N\} = \{e_1, e_2, \ldots, e_{N+1-M}\},$$

$$d'_j \stackrel{\text{def}}{=} d_j - 2h - 1 \geq 0, \quad \bar{d}'_j \stackrel{\text{def}}{=} N - d'_j \geq 0, \quad \bar{\boldsymbol{\lambda}} \stackrel{\text{def}}{=} \boldsymbol{\lambda} - (N+1)\boldsymbol{\delta}. \tag{4.37}$$

Then we have

$$\bar{\Xi}_{\mathcal{D}}(\eta; \boldsymbol{\lambda}) \propto \bar{\Xi}_{\bar{\mathcal{D}}}(\eta; \bar{\boldsymbol{\lambda}}), \tag{4.38}$$

whose degree is $\ell_{\bar{\mathcal{D}}} = \ell_{\mathcal{D}} - (2h+1)M = \sum_{j=1}^M d'_j - \frac{1}{2}M(M-1) < \ell_{\mathcal{D}}$. The relation (4.38) implies the equivalence of the two deformed systems; the Darboux-Crum transformations in terms of $\mathcal{D}$ pseudo virtual states $\tilde{\phi}^{\text{os}}_{d_j}$ and the Krein-Adler transformations in terms of $\bar{\mathcal{D}}$ eigenstates ϕ_{e_j} with shifted parameters $\bar{\boldsymbol{\lambda}}$, as explained in detail in [42]. The situation is similar for (s) with a half odd integer h. The degree reduction of the polynomial is realised by factorisation:

$$P_{\text{v}}(\eta; \boldsymbol{\lambda}) = (1 + \eta^2)^{h+\frac{1}{2}} \times \big(\text{a polynomial of degree v} - 2h - 1 \text{ in } \eta\big) \quad (\text{v} > 2h),$$

and the above equivalence holds with a slight modification on the left hand side:

$$(1 + \eta^2)^{-(h+\frac{1}{2})M}\bar{\Xi}_{\mathcal{D}}(\eta; \boldsymbol{\lambda}) \propto \bar{\Xi}_{\bar{\mathcal{D}}}(\eta; \bar{\boldsymbol{\lambda}}). \tag{4.39}$$

For an integer h $(h = m)$, the polynomial $P_{\text{v}}(\eta; \boldsymbol{\lambda})$ $(\text{v} > 2h)$ vanishes but the limit $\lim_{h \to m}(h - m)^{-1}P_{\text{v}}(\eta; \boldsymbol{\lambda})$ exists and its degree is v. Thus it is not divisible by a non-polynomial factor $(1 + \eta^2)^{m+\frac{1}{2}}$. The degree reduction (4.36) does not occur for (hst), too. Thus the above equivalence does not hold for (s) with an integer h and (hst).

5   Summary and Comments

For iso-spectral rational extensions of shape-invariant potentials in one dimensional quantum mechanics, virtual state wavefunctions are essential. For the radial oscillator potentials and the Darboux-Pöschl-Teller potentials and various examples from the discrete quantum mechanics, the virtual state wavefunctions are obtained by twisting parameters based on discrete symmetries of the Hamiltonian [15, 17, 21, 29, 4, 7, 10, 42]. For six shape-invariant potentials having finitely many discrete eigenstates, the concept of overshoot eigenfunctions

is introduced for searching new types of virtual and pseudo virtual wavefunctions. The new type of virtual state wavefunctions exist for (M), (RM), (Kh) [49] and (hDPT). As a byproduct, a new type of pseudo virtual state wavefunctions are discovered for (s), (RM) and (hst).

Acknowledgements

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